Parametrizations of algebraic structures

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Examples of parametrizations of certain types of algebras, where the products are replaced by a family of products indexed by a set Ω, maybe with an algebraic structure and similar axioms:

Associative algebras : $(x * y) * z = x * (y * z)$.

Matching associative algebras (Pirashvili ; Zhang, Gao, Guo): Ω is a set.

$$(x *_\alpha y) *_\beta z = x *_\alpha (y *_\beta z).$$

Family associative algebras (Zhang, Gao): $(\Omega, \star)$ is a semigroup.

$$(x *_\alpha y) *_{\alpha\star\beta} z = x *_\alpha (y *_\beta z).$$

Associative algebras : $(x * y) * z = x * (y * z)$.

ComTriAs algebras (Loday): two products $\cdot$ and $\star$.

$$\begin{aligned}
(x \cdot y) \cdot z &= x \cdot (y \cdot z),\\
(x \star y) \star z &= x \star (y \star z),\\
(x \star y) \star z &= x \star (y \cdot z),\\
(x \cdot y) \star z &= x \cdot (y \star z).
\end{aligned}$$

and another relation: $x \cdot y = y \cdot x$.

Associative algebras : $(x * y) * z = x * (y * z)$.

Diassociative algebras (Loday): two products $\dashv$ and $\vdash$.

$$\begin{aligned}
(x \dashv y) \dashv z &= x \dashv (y \dashv z),\\
(x \dashv y) \dashv z &= x \dashv (y \vdash z),\\
(x \vdash y) \dashv z &= x \vdash (y \dashv z),\\
(x \dashv y) \vdash z &= x \vdash (y \vdash z),\\
(x \vdash y) \vdash z &= x \vdash (y \vdash z).
\end{aligned}$$

Definition

Let $(\Omega, \rightarrow, \triangleright)$ be a set with two products. An Ω-associative algebra has a family $(*_\alpha)_{\alpha\in\Omega}$ of bilinear products such that

$$(x *_{\alpha\triangleright\beta} y) *_{\alpha\rightarrow\beta} z = x *_\alpha (y *_\beta z).$$

If $\alpha \triangleright \beta = \alpha$ and $\alpha \rightarrow \beta = \beta$, this gives Ω-matching associative algebras.
If $\alpha \triangleright \beta = \alpha$ and $\alpha \rightarrow \beta = \alpha \star \beta$, this gives $(\Omega, \star)$-family associative algebras.

This is far too general. We add some constraints. As free associative algebras are tensor algebras, we want that free Ω-associative algebras are based on Ω-typed tensor algebras:

$$T_\Omega(V) = \bigoplus_{n=1}^{\infty} (\mathbb{K}\Omega)^{\otimes(n-1)} \otimes V^{\otimes n}.$$

The tensors of $T_\Omega(V)$ are called Ω-typed words in V.

We define products $(*_\alpha)_{\alpha\in\Omega}$ on $T_\Omega(V)$ by

$$w *_\alpha z = w\cdot\alpha z, \qquad u *_\alpha (v\cdot\beta z) = (u *_{\alpha\triangleright\beta} v)\cdot(\alpha\rightarrow\beta)z,$$

where $u, v, w \in T_\Omega(V)$, $z \in V$ and $\alpha, \beta \in \Omega$. The product $\cdot$ is the concatenation.

If $x_1, x_2, x_3, x_4 \in V$ and $\alpha, \beta, \gamma \in \Omega$:

$$\begin{aligned}
\alpha x_1 x_2 *_\beta x_3 &= \alpha\beta x_1 x_2 x_3,\\
x_1 *_\alpha \beta x_2 x_3 &= (\alpha\triangleright\beta)(\alpha\rightarrow\beta) x_1 x_2 x_3,\\
\alpha x_1 x_2 *_\beta \gamma x_3 x_4 &= (\alpha x_1 x_2 *_{\beta\triangleright\gamma} x_3)\cdot(\beta\rightarrow\gamma)x_4\\
&= (\alpha\triangleright(\beta\triangleright\gamma))(\alpha\rightarrow(\beta\triangleright\gamma))(\beta\rightarrow\gamma)x_1x_2x_3x_4.
\end{aligned}$$

Why this?

- $T_\Omega(V)$ in order to conserve the combinatorial structure of the associative operad.
- $w *_\alpha z = w \cdot \alpha z$ in order to obtain the generation by V of $T_\Omega(V)$.
- With the preceding item,

$$\begin{aligned} u *_\alpha (v.\beta z) &= u *_\alpha (v *_\beta z) \\ &= (u *_{\alpha \triangleright \beta} v) *_{\alpha \to \beta} z \\ &= (u *_{\alpha \triangleright \beta} v) \cdot (\alpha \to \beta) z. \end{aligned}$$

Theorem

The following are equivalent:

1. There exists a nonzero space V such that $(T_\Omega(V), (*_\alpha)_{\alpha\in\Omega})$ is an Ω-associative algebra.
2. For any vector space V, $(T_\Omega(V), (*_\alpha)_{\alpha\in\Omega})$ is the free Ω-associative algebra generated by Ω.
3. $(\Omega, \rightarrow, \triangleright)$ is an extended associative semigroup (EAS).

Definition

An EAS is a set Ω with two binary products $\rightarrow$ and $\triangleright$ such that

$$\begin{aligned}\alpha \rightarrow (\beta \rightarrow \gamma) &= (\alpha \rightarrow \beta) \rightarrow \gamma,\\ (\alpha \triangleright (\beta \rightarrow \gamma)) \rightarrow (\beta \triangleright \gamma) &= (\alpha \rightarrow \beta) \triangleright \gamma,\\ (\alpha \triangleright (\beta \rightarrow \gamma)) \triangleright (\beta \triangleright \gamma) &= \alpha \triangleright \beta.\end{aligned}$$

Examples

Let Ω be a set. We put

$$\forall \alpha, \beta \in \Omega, \qquad \begin{cases} \alpha \rightarrow \beta = \beta, \\ \alpha \triangleright \beta = \alpha. \end{cases}$$

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS.

This gives Ω-matching associative algebras.

Examples

Let $(\Omega, \star)$ be an associative semigroup. We put

$$\forall \alpha, \beta \in \Omega, \qquad \begin{cases} \alpha \to \beta = \alpha \star \beta, \\ \alpha \triangleright \beta = \alpha. \end{cases}$$

Then $(\Omega, \to, \triangleright)$ is an EAS.

This gives $(\Omega, \star)$-family associative algebras.

Examples

Let $(\Omega, \star)$ be a group. We put

$$\forall \alpha, \beta \in \Omega, \qquad \begin{cases} \alpha \rightarrow \beta = \beta, \\ \alpha \triangleright \beta = \alpha \star \beta^{\star -1}. \end{cases}$$

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS.

EAS of cardinality two

Up to an isomorphism, there are 13 EAS of cardinality two, including one which is not related to the preceding examples.

$\rightarrow$	0	1
0	0	0
1	0	1

$\triangleright$	0	1
0	0	0
1	1	0

This EAS will be called the strange EAS.

Any ComTriaAs algebra is an Ω-associative algebra, with Ω given by

$\rightarrow$	0	1
0	0	0
1	0	1

$\triangleright$	0	1
0	0	0
1	1	1

where $\star = *_0$ and $\cdot = *_1$. The underlying EAS is the EAS associated to the semigroup $(\mathbb{Z}/2\mathbb{Z}, \times)$.

If $(A, \dashv, \vdash)$ is a diassociative algebra:

1. $(A, \dashv^{op}, \vdash^{op})$ is an Ω-associative algebra, with the EAS:

$\rightarrow$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\vdash$
$\vdash$	$\vdash$	$\vdash$

$\rhd$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$

or

$\rhd$	$\dashv$	$\vdash$
$\dashv$	$\vdash$	$\dashv$
$\vdash$	$\vdash$	$\vdash$

2. $(A, \dashv, \vdash)$ is an Ω-associative algebra, with the EAS:

$\rightarrow$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\dashv$	$\vdash$

$\rhd$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\dashv$

or

$\rhd$	$\dashv$	$\vdash$
$\dashv$	$\dashv$	$\dashv$
$\vdash$	$\vdash$	$\vdash$

The underlying EAS are the EAS associated to the semigroup $(\mathbb{Z}/2\mathbb{Z}, \times)$ and the strange EAS.

For any EAS Ω, we obtained a combinatorial description of the operad $\mathcal{P}_\Omega$ of Ω-associative algebras, with

$$\mathcal{P}_\Omega(n) = \mathbb{K}\Omega^{n-1} \times \mathfrak{S}_n.$$

Unfortunately, the Koszul dual of $\mathcal{P}_\Omega$ is not necessarily of the form $\mathcal{P}_{\Omega'}$. A notion of linear EAS is required.

Lemma

Let $(\Omega, \rightarrow, \triangleright)$ be a set with two binary operations. We consider the maps

$$\phi : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightarrow \beta, \alpha \triangleright \beta), \end{cases} \qquad c : \begin{cases} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\beta, \alpha). \end{cases}$$

Then $(\Omega, \rightarrow, \triangleright)$ is an EAS if, and only if

$$(\mathrm{Id} \times \phi) \circ (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times \phi) = (\phi \times \mathrm{Id}) \circ (\mathrm{Id} \times c) \circ (\phi \times \mathrm{Id}).$$

Definition

A linear extended associative semigroup (ℓEAS) is a pair (A, Φ) with $\Phi : A \otimes A \longrightarrow A \otimes A$, such that

$$(\mathrm{Id} \otimes \Phi) \circ (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Phi) = (\Phi \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \tau) \circ (\Phi \otimes \mathrm{Id}),$$

where $\tau : A \otimes A \longrightarrow A \otimes A$ is the usual flip:

$$\tau : \begin{cases} A \otimes A & \longrightarrow & A \otimes A \\ a \otimes b & \longrightarrow & b \otimes a. \end{cases}$$

We shall say that (A, Φ) is nondegenerate if Φ is invertible. If so, (A, Φ^{-1}) is also an ℓEAS.

If (A, Φ) is a finite-dimensional ℓEAS, then (A^*, Φ^*) is an ℓEAS.

Examples of ℓEAS

- If $(\Omega, \rightarrow, \triangleright)$ is an EAS, then $\mathbb{K}\Omega$ is an ℓEAS with

$$\Phi(\alpha \otimes \beta) = \alpha \rightarrow \beta \otimes \alpha \triangleright \beta.$$

- If (A, m, Δ) is a bialgebra, then it is an ℓEAS with

$$\Phi(a \otimes b) = \sum a^{(1)} b \otimes a^{(2)}.$$

- If (A, m, Δ) is a Hopf algebra of antipode S, then it is an ℓEAS with

$$\Phi(a \otimes b) = \sum b^{(1)} \otimes S\left(b^{(2)}\right) a.$$

Examples of ℓEAS

This defines a 2-dimensional ℓEAS:

$$\begin{aligned}
\Phi(x \otimes x) &= x \otimes x,\\
\Phi(x \otimes y) &= y \otimes x,\\
\Phi(y \otimes x) &= x \otimes x - x \otimes y - y \otimes x + 2y \otimes y,\\
\Phi(y \otimes y) &= y \otimes y.
\end{aligned}$$

Similarly with the discrete case, for any ℓEAS (A, Φ), we can define a notion of Φ-associative algebra, with similar results.

$$\sum (x *_{b'} y) *_{a'} z = x *_a (y *_b z),$$

with $\Phi(a \otimes b) = \sum b' \otimes a'$.

Theorem

Let (A, Φ) be a finite-dimensional ℓEAS. The Koszul dual of the operad $\mathcal{P}_\Phi$ of Φ-associative algebras is the operad of Φ^*-associative algebras. Moreover, these operads are Koszul.

Let us consider eigenvectors of Φ:

Lemma

Let (A, Φ) be an ℓEAS and let $x \in A$, nonzero, such that $\Phi(x \otimes x) = \lambda x \otimes x$ for $\lambda \in \mathbb{K}$. Then $\lambda = 0$ or 1.

Proposition

The associative products in $\mathcal{P}_\Phi$ are the products $*_a$, with $\Phi(a \otimes a) = a \otimes a$, and their opposites.

If $\Phi(a \otimes a) = 0$, then for any $x, y, z \in A$,

$$x *_a (y *_a z) = 0.$$

Let $(\Omega, \rightarrow, \triangleright)$ be a set with two binary products and V be a vector space with a family $(*_\alpha)_{\alpha\in\Omega}$ or bilinear products. We define a product on $V \otimes \mathbb{K}\Omega$ by

$$(v \otimes \alpha) * (w \otimes \beta) = (v *_{\alpha\triangleright\beta} w) \otimes \alpha \rightarrow \beta.$$

Theorem

1. If Ω is an EAS and V is an Ω-associative algebra, then $V \otimes \mathbb{K}\Omega$ is an associative algebra.
2. If Ω is a nondegenerate EAS and $V \otimes \mathbb{K}\Omega$ is an associative algebra, then V is an Ω-associative algebra.
3. Let W be a nonzero vector space. If $T_\Omega(W) \otimes \mathbb{K}\Omega$ is an associative algebra, then Ω is an EAS.

Let (A, Φ) be a vector space with $\Phi : A \otimes A \longrightarrow A \otimes A$ and V be a vector space with a family $(*_\alpha)_{\alpha \in \Omega}$ or bilinear products. We define a product on $V \otimes A$ by

$$(v \otimes a) * (w \otimes b) = \sum (v *_{a'} w) \otimes b',$$

where $\Phi(a \otimes b) = \sum b' \otimes a'$.

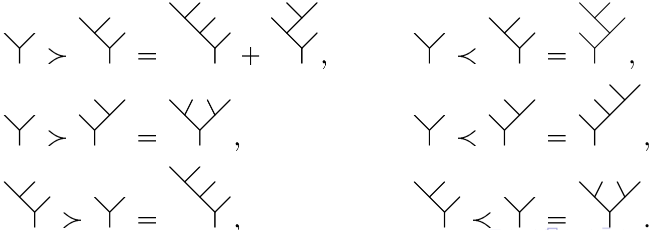
Theorem

1. If (A, Φ) is an ℓEAS and V is a Φ-associative algebra, then $V \otimes A$ is an associative algebra.
2. If (A, Φ) is a nondegenerate ℓEAS and $V \otimes A$ is an associative algebra, then V is a Φ-associative algebra.
3. Let W be a nonzero vector space. If $T_A(W) \otimes A$ is an associative algebra, then (A, Φ) is an ℓEAS.

Dendriform algebras, or noncommutative half-shuffle algebras, are given two products $\prec$ and $\succ$, with

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z),$$
$$(x \succ y) \prec z = x \succ (y \prec z),$$
$$(x \prec y + x \succ y) \succ z = x \succ (y \succ z).$$

Consequently, $\prec + \succ$ is associative. Free dendriform algebras have been described by Loday and Ronco in terms of planar binary trees.



Notions of Ω-matching dendriform algebras and $(\Omega, \star)$-family dendriform algebras have been introduced by Zhang, Gao, Guo and Manchon. Generalizing this:

Definition

Let $(\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ be a set with four binary operations. An Ω-dendriform algebra has two families of products $(\prec_\alpha)_{\alpha\in\Omega}$ and $(\succ_\alpha)_{\alpha\in\Omega}$ such that:

$$(x \prec_\alpha y) \prec_\beta z = x \prec_{\alpha\leftarrow\beta} (y \prec_{\alpha\triangleleft\beta} z) + x \prec_{\alpha\rightarrow\beta} (y \succ_{\alpha\triangleright\beta} z),$$
$$x \succ_\alpha (y \prec_\beta z) = (x \succ_\alpha y) \prec_\beta z,$$
$$x \succ_\alpha (y \succ_\beta z) = (x \succ_{\alpha\triangleright\beta} y) \succ_{\alpha\rightarrow\beta} z + (x \prec_{\alpha\triangleleft\beta} y) \succ_{\alpha\leftarrow\beta} z.$$

Other possibilities are studied by Aguiar.

We wish that free objects are based on Ω-typed planar binary trees, that is to say planar binary trees which internal edges are decorated by elements of Ω. We inductively define products $\prec_\alpha$ and $\succ_\alpha$ on these objects.

Theorem

The following are equivalent:

1. The products on Ω-typed trees define a structure of Ω-dendriform algebra.
2. The products on Ω-typed trees define a structure of Ω-dendriform algebra, freely generated by .
3. Ω is an extended diassociative semigroup (EDS).

Definition

An extended diassociative semigroup Ω has four binary products $\leftarrow$, $\rightarrow$, $\triangleleft$, $\triangleright$ such that

$$(\alpha \leftarrow \beta) \leftarrow \gamma = \alpha \leftarrow (\beta \leftarrow \gamma) = \alpha \leftarrow (\beta \rightarrow \gamma),$$
$$(\alpha \rightarrow \beta) \leftarrow \gamma = \alpha \rightarrow (\beta \leftarrow \gamma),$$
$$(\alpha \rightarrow \beta) \rightarrow \gamma = (\alpha \leftarrow \beta) \rightarrow \gamma = \alpha \rightarrow (\beta \rightarrow \gamma).$$

plus ten other relations involving $\triangleleft$ and $\triangleright$.

Examples

If Ω is a set, put

$$\begin{aligned} \alpha \leftarrow \beta &= \alpha, & \alpha \rightarrow \beta &= \beta, \\ \alpha \triangleleft \beta &= \beta, & \alpha \triangleright \beta &= \alpha. \end{aligned}$$

This defines an EDS, underlying Ω-matching dendriform matching algebras.

Examples

If $(\Omega, \leftarrow, \rightarrow)$ is a diassociative semigroup, put

$$\alpha \triangleleft \beta = \beta, \qquad \alpha \triangleright \beta = \alpha.$$

This defines an EDS. In the particular case where $\rightarrow = \leftarrow = \star$, associative product, this EDS underlies $(\Omega, \star)$-family dendriform algebras.

Examples

Up to isomorphism, there are 24 EDS of cardinality 2.

Let $\Omega = (\Omega, \leftarrow, \rightarrow, \triangleleft, \triangleright)$ be a set with four products. We put

$$\varphi_{\leftarrow} : \left\{ \begin{array}{rcl} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \leftarrow \beta, \alpha \triangleleft \beta), \end{array} \right.$$

$$\varphi_{\rightarrow} : \left\{ \begin{array}{rcl} \Omega^2 & \longrightarrow & \Omega^2 \\ (\alpha, \beta) & \longrightarrow & (\alpha \rightarrow \beta, \alpha \triangleright \beta). \end{array} \right.$$

Lemma

Ω is an EDS if, and only if:

$$
\begin{aligned}
&(c \times Id) \circ (Id \times \varphi_{\leftarrow}) \circ (c \times Id) \circ (\varphi_{\rightarrow} \times Id)\\
&= (\varphi_{\rightarrow} \times Id) \circ (Id \times \varphi_{\leftarrow}),\\
&(Id \times \varphi_{\leftarrow}) \circ (c \times Id) \circ (Id \times \varphi_{\leftarrow}) \circ (c \times Id) \circ (\varphi_{\leftarrow} \times Id)\\
&= (\varphi_{\leftarrow} \times Id) \circ (Id \times \varphi_{\leftarrow}),\\
&(Id \times \varphi_{\rightarrow}) \circ (c \times Id) \circ (Id \times \varphi_{\leftarrow}) \circ (c \times Id) \circ (\varphi_{\leftarrow} \times Id)\\
&= (\varphi_{\leftarrow} \times Id) \circ (Id \times \varphi_{\rightarrow}),\\
&(Id \times \varphi_{\leftarrow}) \circ (\varphi_{\rightarrow} \times Id) \circ (Id \times \varphi_{\rightarrow})\\
&= (\varphi_{\rightarrow} \times Id) \circ (Id \times c) \circ (\varphi_{\leftarrow} \times Id),\\
&(Id \times \varphi_{\rightarrow}) \circ (\varphi_{\rightarrow} \times Id) \circ (Id \times \varphi_{\rightarrow})\\
&= (\varphi_{\rightarrow} \times Id) \circ (Id \times c) \circ (\varphi_{\rightarrow} \times Id).
\end{aligned}
$$

This leads to a notion of linear EDS and to related generalized dendriform algebras. Within this frame, it is possible to describe the Koszul dual of the operad of Ω-dendriform algebras (giving generalizations of diassociative algebras) and to prove that, if Ω is non degenerate, then the operad of Ω-dendriform algebras is Koszul.

An Ω-pre-Lie algebra satisfies

$$x \circ_\alpha (y \circ_\beta z) - (x \circ_{\alpha \triangleright \beta} y) \circ_{\alpha \rightarrow \beta} z = y \circ_\beta (x \circ_\alpha z) - (y \circ_{\beta \triangleright \alpha} x) \circ_{\beta \rightarrow \alpha} z.$$

We obtain similar results for these objects. The structure on Ω has to be a commutative EDS, that is to say an EDS such that, for any $\alpha, \beta \in \Omega$,

$$\alpha \leftarrow \beta = \beta \rightarrow \alpha, \qquad \alpha \triangleleft \beta = \beta \triangleright \alpha.$$

These objects give a way to obtain double bialgebras on typed rooted trees.

An Ω-Rota-Baxter algebra satisfies

$$P_\alpha(a)P_\beta(b) = P_{\alpha\rightarrow\beta}(P_{\alpha\triangleright\beta}(a)b) + P_{\alpha\leftarrow\beta}(aP_{\alpha\triangleleft\beta}(b)) + \lambda_{\alpha,\beta}P_{\alpha\cdot\beta}(ab).$$

(Five products on Ω and a family of scalars indexed by Ω^2). This leads to the notion of λ-triassociative semigroups, closely related to the notion of EDS.

An Ω-tridendriform algebra satisfies

$$\begin{aligned}
(a \prec_\alpha b) \prec_\beta c =&\ a \prec_{\alpha\rightarrow\beta} (b \succ_{\alpha\triangleright\beta} c) + a \prec_{\alpha\leftarrow\beta} (b \prec_{\alpha\triangleleft\beta} c)\\
&+ a \prec_{\alpha\cdot\beta} (b \circ_{\alpha*\beta} c),\\
(a \succ_\alpha b) \prec_\beta c =&\ a \succ_\alpha (b \prec_\beta c),\\
a \succ_\alpha (b \succ_\beta c) =&\ (a \succ_{\alpha\triangleright\beta} b) \succ_{\alpha\rightarrow\beta} c + (a \prec_{\alpha\triangleleft\beta} b) \succ_{\alpha\leftarrow\beta} c\\
&+ (a \circ_{\alpha*\beta} b) \succ_{\alpha\cdot\beta} c,\\
(a \succ_\alpha b) \circ_\beta c =&\ a \succ_\alpha (b \circ_\beta c),\\
(a \prec_\alpha b) \circ_\beta c =&\ a \circ_\beta (b \succ_\alpha c),\\
(a \circ_\alpha b) \prec_\beta c =&\ a \circ_\alpha (b \prec_\beta c,)\\
(a \circ_\alpha b) \circ_\beta c =&\ a \circ_\alpha (b \circ_\beta c).
\end{aligned}$$

This leads to the notion of extended triassociative semigroup (six products).

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Thank you for your attention!