On the tensor product of cocomplete quantale-enriched categories

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Motivation

- Universal coalgebra as a uniform framework for state-based transition systems: from behavioural equivalence to behavioural distance [14, 20]
- Quantitative domain theory: algebraic structure with prominent features of convergence and approximation [1, 3, 6, 15]
- Many-valued formal concept analysis: extracting information from contexts with quantitative data [19]

Keywords

- Quantales
- Quantale-enriched categories
- Tensor product & related

  > It is often the case that **the closed structure** is **primary** and **the tensor product** is defined as a left adjoint to it [...], but its construction is much less intuitive [...] (and) gives little information about what it actually looks like. (nLab)

- Cocomplete quantale-enriched categories
- Completely distributive cocomplete quantale-enriched categories

Quantales

A commutative quantale $\mathcal{V}$ is a commutative monoid in SupLat:

$(\mathcal{V}, \vee, \perp)$ is a complete sup-lattice

$(\mathcal{V}, \otimes, e)$ is a commutative monoid such that
$- \otimes -$ preserves arbitrary sups

Consequence: every $- \otimes v : \mathcal{V} \to \mathcal{V}$ has a right adjoint $[v, -] : \mathcal{V} \to \mathcal{V}$

$$u \otimes v \leq w \iff u \leq [v, w] \qquad \text{modus ponens}$$

Examples

- $(2, \wedge, 1)$
- $([0, \infty]^{op}, +, 0)$
- $([0,1], \otimes, 1)$, with $\otimes$ the usual product/min/Łukasiewicz product
- The quantale of left continuous distribution functions
  $\Delta = \{f : [0,\infty] \to [0,1] \mid f(a) = \bigvee_{b<a} f(b)\}$

Quantale-enriched categories

Let $\mathcal{V}$ be a commutative quantale. A $\mathcal{V}$-enriched category $\mathcal{A}$ consists of a set of objects, together with a $\mathcal{V}$-valued relation (usually called $\mathcal{V}$-hom, or $\mathcal{V}$-distance, or $\mathcal{V}$-metric)

$$\mathcal{A}(-,-) : \mathcal{A} \times \mathcal{A} \to \mathcal{V}$$

satisfying

$$e \leq \mathcal{A}(a,a) \quad \text{and} \quad \mathcal{A}(a,b) \otimes \mathcal{A}(b,c) \leq \mathcal{A}(a,c)$$

Examples

- $\mathcal{V} = (2, \wedge, 1) \Longrightarrow \mathcal{V}$-enriched categories are ordered sets
- $\mathcal{V} = ([0,\infty]^{op}, +, 0) \Longrightarrow \mathcal{V}$-enriched categories are (generalised) metric spaces
- $\mathcal{V} = \Delta \Longrightarrow \mathcal{V}$-enriched categories are probabilistic metric spaces [13]

Quantale-enriched categories

Denote by $\mathcal{V}$-cat the (2-)category of $\mathcal{V}$-categories and $\mathcal{V}$-functors.

$\mathcal{V}$-cat is symmetric monoidal closed:

- The tensor product $\mathcal{A} \otimes \mathcal{B}$ of two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ has pairs (a, b) with $a \in \mathcal{A}$, $b \in \mathcal{B}$ as objects, and $\mathcal{V}$-homs

$$(\mathcal{A} \otimes \mathcal{B})((a', b'), (a, b)) = \mathcal{A}(a', a) \otimes \mathcal{B}(b', b)$$

- The unit for the tensor product is the $\mathcal{V}$-category $\mathbb{1}$, with one object 0 and corresponding $\mathcal{V}$-hom given by $\mathbb{1}(0, 0) = e$.

- The internal hom between two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$ is the $\mathcal{V}$-category of $\mathcal{V}$-functors $\mathcal{A} \to \mathcal{B}$ with “uniform” $\mathcal{V}$-distances

$$[\mathcal{A}, \mathcal{B}](f, g) = \bigwedge_a \mathcal{B}(fa, ga)$$

Free cocompletion monad

- Denote $\mathbb{D}\mathcal{A} = [\mathcal{A}^{op}, \mathcal{V}]$. The correspondence $\mathcal{A} \mapsto \mathbb{D}\mathcal{A}$ produces a monad
$$\mathbb{D} : \mathcal{V}\text{-cat} \to \mathcal{V}\text{-cat}$$
with unit the Yoneda embedding
$$y_{\mathcal{A}} : \mathcal{A} \to \mathbb{D}\mathcal{A}\,,\ y(a) = \mathcal{A}(-, a)$$
and multiplication the $\mathcal{V}$-"union" of downsets.

- $\mathbb{D}$ is the free cocompletion monad on $\mathcal{V}$-cat; in particular, it is a Kock-Zöberlein-monad.

- A $\mathbb{D}$-algebra is a cocomplete $\mathcal{V}$-category $\mathcal{A}$, with structure provided by the left adjoint $\sup_{\mathcal{A}}$ of $y_{\mathcal{A}}$

$$\mathcal{A} \underset{y_{\mathcal{A}}}{\overset{\sup_{\mathcal{A}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{D}\mathcal{A}$$

Multiple facets of cocomplete $\mathcal{V}$-categories [16, 18]

- Algebras for the free cocompletion monad $\mathbb{D}$ on $\mathcal{V}$-cat

- Injective $\mathcal{V}$-enriched categories (wrt fully faithful $\mathcal{V}$-functors)

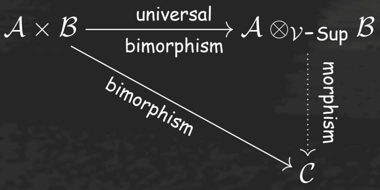
- Modules over the monoid $\mathcal{V}$ within the category of sup-lattices

- Algebras for the $\mathcal{V}$-valued powerset monad on Set (separated $\mathcal{V}$-Sup)

Tensor product of cocomplete $\mathcal{V}$-categories

- Denote by $\mathcal{V}$-Sup the category of cocomplete $\mathcal{V}$-categories and cocontinuous $\mathcal{V}$-functors (category of $\mathbb{D}$-algebras).

- $\mathbb{D}$ is a commutative monad, therefore $\mathcal{V}$-Sup is symmetric monoidal closed:
  - The tensor product $\otimes_{\mathcal{V}\text{-Sup}}$ classifies bimorphisms [12]



  - The unit is $\mathbb{D}\mathbb{1} = \mathcal{V}$
  - The internal hom is $\mathcal{V}\text{-Sup}(\mathcal{A}, \mathcal{B})$.

Tensor product of cocomplete $\mathcal{V}$-categories

- The inverter

$$\mathcal{A} \otimes_{\mathcal{V}\text{-Sup}} \mathcal{B} \hookrightarrow \mathbb{D}(\mathcal{A} \otimes \mathcal{B}) \underset{\Downarrow}{\rightrightarrows} \mathbb{D}(\mathbb{D}\mathcal{A} \otimes \mathbb{D}\mathcal{B})$$

  exhibits $\mathcal{A} \otimes_{\mathcal{V}\text{-Sup}} \mathcal{B}$ as reflective in $\mathbb{D}(\mathcal{A} \otimes \mathcal{B})$ [2]

- There is a duality $\mathcal{V}\text{-Sup} \cong \mathcal{V}\text{-Sup}^{op}$, sending $\mathcal{A}$ to $\mathcal{A}^{op}$ and $f : \mathcal{A} \to \mathcal{B}$ to $g^{op} : \mathcal{B}^{op} \to \mathcal{A}^{op}$, where $f \dashv g$

- In particular, $\mathcal{A}^{op} \cong \mathcal{V}\text{-Sup}(\mathcal{V}, \mathcal{A}^{op}) \cong \mathcal{V}\text{-Sup}(\mathcal{A}, \mathcal{V}^{op})$

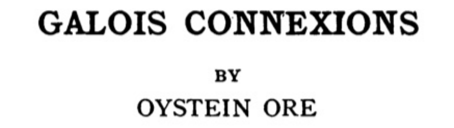
- This implies that $\mathcal{V}$-Sup is a **$*$-autonomous** category, with dualizer $\mathcal{V}^{op}$ [5]

$\mathcal{V}$-"Sup is good food" (R. Blute, FMCS 2022)

Tensor product of cocomplete $\mathcal{V}$-categories

- Consequently, the tensor product can be equivalently described using Galois connections [2, 5]

$$\mathcal{A} \otimes_{\mathcal{V}\text{-Sup}} \mathcal{B} \cong \mathcal{V}\text{-Sup}(\mathcal{A}, \mathcal{B}^{\mathrm{op}})^{\mathrm{op}}$$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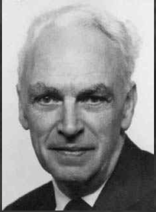
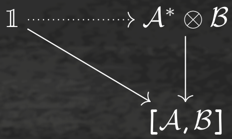


- What about other (monoidal) features of $\mathcal{V}$-Sup?

Nuclearity

- Grothendieck introduced in Functional Analysis the concept of **nuclearity** for objects and morphisms, in order to mimic finite dimensionality behaviour (for objects) and matrix calculus (for arrows) [8]

- It was later realised that nuclearity can be defined in the more general context of (symmetric) monoidal closed categories:
  - An arrow f : $\mathcal{A} \to \mathcal{B}$ is nuclear iff the associated $\mathbb{1} \to [\mathcal{A}, \mathcal{B}]$ factorises through $\mathcal{A}^* \otimes \mathcal{B}$, where $\mathcal{A}^* = [\mathcal{A}, \mathbb{1}]$:

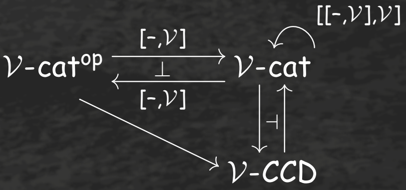


  - An object $\mathcal{A}$ is nuclear (dualizable) iff $id_{\mathcal{A}}$ is so. [9]

Nuclearity – examples

- For any $\mathbb{k}$-associative algebra A, the category Mod_A of right A-modules is nuclear in the (2-)category $\mathrm{LocPres}_{\mathbb{k}}$ of locally presentable $\mathbb{k}$-linear categories and cocontinuous $\mathbb{k}$-linear functors, wrt the Kelly-Deligne tensor product $\boxtimes$ (follows from the Eilenberg-Watts thm)
- Let C be a $\mathbb{k}$-coassociative coalgebra. Then the category Comod^C of right C-comodules is nuclear in $\mathrm{LocPres}_{\mathbb{k}}$ if and only if it has enough projectives [4]
- Nuclear objects in SupLat are the completely distributive lattices [9]
- What about in $\mathcal{V}$-Sup?

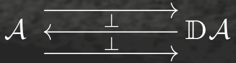
Completely distributive $\mathcal{V}$-categories



- The algebras for the double dualization monad $[-,\mathcal{V}],\mathcal{V}]$ on $\mathcal{V}$-cat: completely distributive $\mathcal{V}$-categories ($\mathcal{V}$-CCD) [10, 18]

  Homomorphisms: continuous and cocontinuous $\mathcal{V}$-functors.

- Equivalently, a $\mathcal{V}$-category $\mathcal{A}$ is $\mathcal{V}$-CCD iff the Yoneda embedding $y_{\mathcal{A}}$ has a left adjoint $\sup_{\mathcal{A}}$ ($\mathcal{A}$ is cocomplete) which has also a left adjoint $\Downarrow_{\mathcal{A}}$ [17]



- Yet another description: $\mathcal{V}$-CCD are the projective objects of $\mathcal{V}$-Sup [17]

Tensor product of $\mathcal{V}$-CCD in $\mathcal{V}$-Sup [2]

- For $\mathcal{A}, \mathcal{B} \in \mathcal{V}$-CCD, $\mathcal{A} \otimes_{\mathcal{V}\text{-Sup}} \mathcal{B}$ is again $\mathcal{V}$-CCD

- Using the split idempotent completion of the category of $\mathcal{V}$-categories and $\mathcal{V}$-distributors, it follows that every $\mathcal{V}$-CCD is nuclear in $\mathcal{V}$-Sup

- Conversely, each nuclear object in $\mathcal{V}$-Sup is projective, hence $\mathcal{V}$-CCD

- What about the symmetric monoidal closed category $\mathcal{V}$-CCD (with continuous and cocontinuous $\mathcal{V}$-functors as arrows)?

Addendum: the Isbell completion

- For each $\mathcal{V}$-category $\mathcal{A}$, there is an adjunction commuting with the Yoneda embeddings

$$\begin{array}{ccc} & \mathcal{A} & \\ {\scriptstyle \mathsf{Y}_{\mathcal{A}}} \swarrow & & \searrow {\scriptstyle \mathsf{Y}'_{\mathcal{A}}} \\ [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] & \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} & [\mathcal{A}, \mathcal{V}]^{\mathrm{op}} \end{array}$$

- The fixed points of this adjunction determine a complete and cocomplete $\mathcal{V}$-category $\mathrm{I}\mathcal{A}$ into which $\mathcal{A}$ embeds, known as the Isbell completion of a $\mathcal{V}$-category (the categorical analogue of the Dedekind-MacNeille completion of a poset) [11]

- $\mathcal{V}$-Sup is reflective in the category of $\mathcal{V}$-categories and cut-cocontinuous $\mathcal{V}$-functors, with reflection given by the Isbell completion [7]

Addendum: the Isbell completion

- When is the Isbell completion of a $\mathcal{V}$-category $\mathcal{V}$-CCD? It seems unlikely that a good explicit description could be achieved in general (for $\mathcal{V} = \mathbb{2}$ it involves taking complements of relations)

- However, $\mathcal{V}$ is a Girard quantale, then the Isbell completion of a $\mathcal{V}$-category $\mathcal{A}$ is $\mathcal{V}$-CCD iff the "negation" of the $\mathcal{V}$-hom distributor $\mathcal{A}(-,-)$ is a **regular** $\mathcal{V}$-relation [2]

Conclusions

M. Fréchet

Sur quelques points du calcul fonctionnel (1906)

I1 fallait d'abord voir comment transformer les énoncés des théorèmes pour qu'ils conservent un sens dans le cas général. I1 fallait ensuite, soit transcrire les démonstrations dans un langage plus général, soit, lorsque cela n'était pas possible, donner des démonstrations nouvelles et plus générales. Il s'est trouvé que les démonstrations que nous avons ainsi obtenues sont souvent aussi simples, et quelquefois même plus simples, que les démonstrations particulières qu'elles remplacaient. Cela tient sans doute à ce que la position de la question obligeait à ne faire usage que de ses particularités vraiment essentielles.

Thank you for your attention!

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